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THE AMALGAMATION PROPERTY AND A PROBLEM OF HENKIN, MONK AND TARSKI

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Abstract

Using the fact that the class of representable cylindric algebras of infinite dimension fails to have the amalgamation property, we solve an open problem in the monograph "Cylindric Algebras, Part I" by Henkin, Monk and Tarski. Our result applies to other algebras of logic, namely Pinter's substitution algebras and Halmo's quasi-polyadic algebras.

1. Introduction

Problem 2.13 on p. 464 of [6] asks whether every representable cylindric algebra of infinite dimension satisfies a certain property formulated in terms of reducts of the algebra in question which is (iii) in Theorem 2.6.50 in [6]. We give a negative answer to this question by showing that the class of algebras satisfying (iii) in Theorem 2.6.50 necessarily have the amalgamation property, while it is known that the class of representable cylindric algebras does not have the amalgamation property. However, we will not only address cylindric algebras, but our

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investigations will cover also other algebras for which this question is meaningful. The concrete versions of algebras considered consist of sets of sequences, and the operations are set-theoretic operations on such sets. Let α be an ordinal. Let U be a set. Then, we define for $i, j < \alpha$ and $X \subseteq {}^{\alpha}U$:

$$c_i X = \{s \in {}^{\alpha}U : \exists t \in X, t(j) = s(j) \text{ for all } j \neq i\},$$

$$s_i^j X = \{s \in {}^{\alpha}U : s \circ [i|j] \in X\},$$

$$s_{[i,j]} X = \{s \in {}^{\alpha}U : s \circ [i, j] \in X\},$$

$$d_{ij} = \{s \in {}^{\alpha}U : s_i = s_j\}$$

$[\,|\,]$ is the replacement on α that takes i to j and leaves every other thing fixed, while $[i, j]$ is the transposition interchanging i and j. The extra non-boolean operations we deal with are as specified above. For set X, let $\mathfrak{B}(X) = \langle \wp(X), \cup, \cap, \sim, \emptyset, X \rangle$ be the full boolean set algebra with universe $\wp(X)$. Let S be the operation of forming subalgebras, and P be that of forming products.

Then

$$\mathbf{RSC}_\alpha = SP\{(\mathfrak{B}({}^{\alpha}U), c_i, s_i^j)_{i,j<\alpha} : U \text{ is a set}\}.$$

$$\mathbf{RQA}_\alpha = SP\{(\mathfrak{B}({}^{\alpha}U), c_i, s_i^j, s_{[i,j]})_{i,j<\alpha} : U \text{ is a set}\}.$$

$$\mathbf{RCA}_\alpha = SP\{(\mathfrak{B}({}^{\alpha}U), c_i, d_{ij})_{i,j<\alpha} : U \text{ is a set}\}.$$

$$\mathbf{RQEA}_\alpha = SP\{(\mathfrak{B}({}^{\alpha}U), c_i, d_{ij}, s_{[i,j]})_{i,j<\alpha} : U \text{ is a set}\}.$$

$\mathbf{SC}$ stands for the class of Pinter's substitution algebras. $\mathbf{QA}$ ($\mathbf{QEA}$) for quasi-polyadic (equality) algebras and $\mathbf{CA}$ stands for the class of cylindric algebras. In our treatment of cylindric algebras, we follow the notation and terminology of [6], while for $\mathbf{QA}$'s and $\mathbf{QEA}$'s we follow [13] and for $\mathbf{SC}$'s we follow [10], see also [11]. These are abstract classes defined by a finite schema of equations in the same similarity type of the representable algebras defined above.

Definition 1. An algebra in $\mathbf{SC}_\alpha$ is of the form

$$\mathfrak{A} = (A, +, \cdot, -, 0, 1, c_i, s_i^j)_{i, j \in \alpha},$$

where $(A, +, \cdot, -, 0, 1)$ is a boolean algebra c_i, s_i^j are unary operations on $\mathfrak{A}(i, j < \alpha)$ satisfying the following equations for all $i, j, k, l \in \alpha$

1. $c_j 0 = 0$, $x \leq c_i x$, $c_i(x \cdot c_i y) = c_i x \cdot c_i y$, and $c_i c_j x = c_j c_i x$,

2. $s_i^i x = x$,

3. s_j^i are boolean endomorphisms,

4. $s_j^i c_i x = c_i x$,

5. $c_i s_j^i x = s_j^i x$, whenever $i \neq j$,

6. $s_j^i c_k x = c_k s_j^i x$, whenever $k \notin \{i, j\}$,

7. $c_i s_i^j x = c_j s_j^i x$.

All algebras considered have an $\mathbf{SC}$ reduct. Let us get back to Problem 2.13 in [6]. Item (iv) in Theorem 2.6.50 speaks rather of the class $S\mathfrak{Nr}_\alpha \mathbf{CA}_{\alpha+w}$, referred as the class of algebras having the neat embedding property. For this reason, we need to review the notion of neat reducts. The reader is refered to [1] for an overview. We concentrate on cylindric algebras (for the time being). If $\mathfrak{C} \in \mathbf{Cs}_\beta$, (i.e., $\mathfrak{C}$ is a cylindric set algebra of dimension β) with unit ${}^\beta U$, then for any $\alpha < \beta$, the elements of $\mathfrak{C}$ that are fixed by c_i, $i \geq \alpha$, can be thought of as representations of α ary relations on U. In fact, if we keep only these elements and those operations whose indices are all in α, then the resulting algebra is obviously isomorphic to a $\mathbf{Cs}_\alpha$ (and in fact to one with base U). This observation carries over to abstract $\mathbf{CA}_\beta$'s in general yielding the concept of neat reducts. A reduct of an algebra $\mathfrak{A}$ is another algebra $\mathfrak{B}$ obtained from $\mathfrak{A}$ by dropping some of the operations. $\mathfrak{B}$ thus has the same universe of $\mathfrak{A}$ but the operations defined on these elements are only some of the original operations. In cylindric algebras, reducts are important because certain reducts of cylindric algebras are cylindric algebras (of a different dimension, though). Let $\mathfrak{A} = (A, +, \cdot, -, c_i, d_{ij}) \in \mathbf{CA}_\beta$ and $\rho : \alpha \to \beta$ be one to one. Then $\mathfrak{Rd}^\rho\mathfrak{A} = (A, +, \cdot, -, c_{\rho(i)}, d_{\rho(i), \rho(j)})_{i, j<\alpha}$ is a $\mathbf{CA}_\alpha$. Here a reduct is defined by renaming the operations. When $\alpha \subseteq \beta$ and ρ is the inclusion map, then $\mathfrak{Rd}_\alpha\mathfrak{A}$ is just the algebra obtained by discarding the operations indexed by ordinals in $\beta \sim \alpha$. For $x \in \mathfrak{A}$, let $\Delta x = \{i \in \beta : c_i x \neq x\}$. Then for $i, j < \alpha$ we have $\Delta d_{ij} \subseteq \alpha$ and if $\Delta x \subseteq \alpha$ and $i < \alpha$, then $\Delta c_i x \subseteq \alpha$. By noting that $\Delta(x + y) \subseteq \Delta x \cup \Delta y$ and $\Delta(-x) = \Delta x$, if we take the set $Nr_\alpha\mathfrak{B} = \{x \in B : \Delta x \subseteq \alpha\}$, then this set is a subuniverse of $\mathfrak{Rd}_\alpha\mathfrak{B}$. Then the algebra $\mathfrak{Nr}_\alpha\mathfrak{B} \in \mathbf{CA}_\alpha$ with universe $Nr_\alpha\mathfrak{B}$ is called the neat α reduct of $\mathfrak{B}$. If there is an embedding $e : \mathfrak{C} \to \mathfrak{Nr}_\alpha\mathfrak{B}$, then we say that $\mathfrak{C}$ neatly embeds in $\mathfrak{B}$. For $K \subseteq \mathbf{CA}_\beta$ and $\alpha < \beta$, $\mathfrak{Nr}_\alpha K = \{\mathfrak{Nr}_\alpha\mathfrak{B} : \mathfrak{B} \in K\}$. Now if $\alpha < \beta$, then $\theta : \wp({}^\alpha U) \to \wp({}^\beta U)$ defined by

$$X \mapsto \{s \in {}^\beta U : (s \mid \alpha) \in X\}$$

maps $\wp({}^\alpha U)$ into $\mathfrak{Nr}_\alpha\wp({}^\beta U)$. Thus set algebras can be neatly embedded into algebras in arbitary extra dimensions. But the converse is strikingly true. If $\mathfrak{A} \in \mathbf{CA}_\alpha$ and there exists an embedding $e : \mathfrak{A} \to \mathfrak{Nr}_\alpha\mathfrak{B}$, with $\mathfrak{B} \in \mathbf{CA}_{\alpha+w}$, then $\mathfrak{A}$ is representable. So, we have the following (neat) Neat Embedding Theorem, cf. [7] Theorem 3.2.10, or *NET* for short, of Henkin: $\mathbf{RCA}_\alpha = S\mathfrak{Nr}_\alpha\mathbf{CA}_{\alpha+w}$ for any α. Here S stands for the operation of forming subalgebras. Algebras in the class $S\mathfrak{Nr}_\alpha\mathbf{CA}_{\alpha+w}$ are said to have the Neat Embedding Property (*NEP*). Monk proved that $\mathbf{RCA}_\alpha \subset S\mathfrak{Nr}_\alpha\mathbf{CA}_{\alpha+n}$ for every $\alpha > 2$ and $n \in w$, so that all w extra dimensions are needed to enforce representability, cf [7] Theorem 3.2.85. The famous non-finite axiomatizability result for $\mathbf{RCA}_\alpha$ when $\alpha > 2$, follows from Monk's result. Completely analogous observations carry over to the other algebras dealt with in this paper. In particular, neat reducts are defined completely analogously and we do have a Neat Embedding Theorem for such algebras as well. In what follows, we formulate two theorems that provide sufficient conditions for a class of algebras to have the amalgamation property, and super amalgamation property, respectively. Then we will use such theorems to solve Problem 2.13 in [6] together with other related problems that are scattered in the literature. We start by recalling the notion of the amalgamation property and super amalgamation due to Maksimova [9]. The super amalgamation property has been extensively studied in connection to interpolation and definability in modal and intuitionistic logic by Maksimova, Gabbay [5] and others. In this paper, we apply this notion rather to classical logic.

Definition 2. (1) Let V be a class of algebras (usually but not always assumed to be a variety) and $L_1, L_2 \subseteq V$. L_2 is said to have the amalgamation property, or *AP* for short over L_1, with respect to V, if for all $\mathfrak{A}_0 \in L_1$ all $\mathfrak{A}_1$ and $\mathfrak{A}_2 \in L_2$, and all monomorphisms i_1 and i_2 of $\mathfrak{A}_0$ into $\mathfrak{A}_1, \mathfrak{A}_2$, respectively, there exists $\mathfrak{A} \in V$, a monomorphism m_1 from $\mathfrak{A}_1$ into $\mathfrak{A}$ and a monomorphism m_2 from $\mathfrak{A}_2$ into $\mathfrak{A}$ such that $m_1 \circ i_1 = m_2 \circ i_2$. In this case, we say that $\mathfrak{A}$ is an amalgam of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ over $\mathfrak{A}_0$ via m_1 and m_2 or even simply an amalgam.

(2) We say that L_2 has the strong *AP*, or *SAP* for short over L_1 with respect to V, if in addition to (1), we have $m_1 \circ i_1(A_0) = m_1(A_1) \cap m_2(A_2)$. In this case, we say that $\mathfrak{A}$ is a strong amalgam of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ over $\mathfrak{A}_0$, via m_1 and m_2, or even simply a strong amalgam.

(3) Assume that V is a class of boolean algebras with extra operations. We say that L_2 has *SUPAP* over L_1 with respect to V, if in addition to (1) we have

$$(\forall x \in A_j)(\forall y \in A_k)(m_j(x) \le m_k(y) \Rightarrow (\exists z \in A_0)(x \le i_j(z) \wedge i_k(z) \le y)),$$

where $\{j, k\} = \{1, 2\}$. Here $\leq$ is the boolean order. In this case, we say that $\mathfrak{A}$ is a super amalgam of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ over $\mathfrak{A}_0$, via m_1 and m_2, or even simply a super amalgam.

(4) When $L_1 = L_2 = L$ in (1) above, we say that L has *AP* with respect to *V*. If furthermore $L = V$, we say that *V* simply has *AP*. A similar observation holds for *SAP* and *SUPAP*.

Note that the more conventional definition of *AP*, *SAP* and *SUPAP* is when $L_1 = L_2 = V$. We note that *SUPAP* is stronger than *SAP* for boolean algebras with extra operations, and for (varieties of) cylindric algebras, it is strictly stronger, a result of Sagi and Shelah [14].

From now on we use the notation and terminology of [6] with obvious modification for the other algebras. In particular, $\mathfrak{Rd}_\alpha^\rho\mathfrak{B}$ denotes the ρ reduct of $\mathfrak{B}$ [6], Definition 2.6.1, $\mathfrak{Nr}_\alpha\mathfrak{B}$ denotes the α neat reduct of $\mathfrak{B}$ [6], Definition 2.6.28, and $\mathfrak{Fr}_\beta^\rho L$ stands for the dimension restricted free algebra over L with β generators, dimension restricted by ρ [6], Definition 2.5.31. The sequence $\langle \eta / Cr_\beta^\rho L : \eta < \beta \rangle$ L-freely generates $\mathfrak{Fr}_\beta^\rho L$ [6], Theorem 2.5.35. For $X \subseteq \mathfrak{A}$, $\mathfrak{Sg}^{\mathfrak{A}}X$ or simply $\mathfrak{Sg}X$ denotes the subalgebra of $\mathfrak{A}$ generated by X. $\mathfrak{Ig}^{\mathfrak{A}}X$ stands for the ideal generated by *X*. In the following theorem condition (2) is just the condition 2.6.50 (iii) in [6] extrapolated to other algebras considered herein.

Theorem 3. *Let* $L \subseteq \mathbf{K}_\alpha$. *Consider the following conditions*:

(1) *For all* $\mathfrak{A} \in L$ *for all non-zero x in A, for all finite* $\Gamma \subseteq \alpha$, *there exist distinct* $i, j \in \alpha \setminus \Gamma$, *such that* $s_i^j x \neq 0$.

(2) *For all* $\mathfrak{A} \in L$ *for every finite sequence* ρ *without repeating terms and with range included in* α, *for every non-zero* $x \in A$, *there is a function h and* $k < \alpha$ *such that h is an endomorphism of* $\mathfrak{Rd}^\rho\mathfrak{A}$, $k \in \alpha \setminus Rg\rho$, $c_k \circ h = h$ *and* $h(x) \neq 0$.

(3) (a) *If whenever* $\mathfrak{A} \in L$, *there exists* $x \in^{|A|} A$ *such that if* $\rho = \langle \Delta x_i : i < |A| \rangle$, $\mathfrak{D} = \mathfrak{Fr}^{\rho}_{|A|}\mathbf{K}_{\beta}$ *and* $g_{\xi} = \xi / Cr^{\rho}_{|A|}\mathbf{K}_{\beta}$, *then* $\mathfrak{Sg}^{\mathfrak{Rd}_{\alpha}\mathfrak{D}}\{g_{\xi} : \xi < |A|\} \in L$,

(b) *If* $\mathfrak{A} \in L$, *then for any* $\mathfrak{B} \in \mathbf{K}_{\alpha+w}$ *such that* $\mathfrak{A} \subseteq \mathfrak{Nr}_{\alpha}\mathfrak{B}$ *and A generates* $\mathfrak{B}$, *and for any ideal of I of* $\mathfrak{B}$, *we have* $I \subseteq \mathfrak{Ig}^{\mathfrak{B}}(I \cap A)$,[1]

then in (1) *and* (2) *L has AP with respect to* $\mathbf{RK}_{\alpha}$ *over L, and in* (3) $\mathbf{RK}_{\alpha}$ *has AP with respect to* $\mathbf{RK}_{\alpha}$ *over L.*

Proof. Note that algebras satisfying (2) are representable, cf. [6] Theorem 2.6.50. It is not hard to show that (1) implies (2). Now assume (2). Then the following condition holds for every $\lambda < w$: for every $k < w$ for every one to one $\rho \in {}^{k}\alpha$ and every non-zero $x \in A$, there exist σ, h such that $\sigma \in^{k+\lambda} \alpha$, σ is one to one, $\rho \subseteq \sigma$, h is an endomorphism of $\mathfrak{Rd}^{\rho}_{k}\mathfrak{A}$, $c_{\sigma_u} \circ h = h$, whenever, $k \leq \mu < k + \lambda$ and $h(x) \neq 0$. This can be proved by induction on λ, cf. [6] p. 416. Let $\mathfrak{A}_0$ and $\mathfrak{A}_1 \in L$. We claim that there exist $\mathfrak{B}_0, \mathfrak{B}_1 \in \mathbf{K}_{\alpha+w} j_0 : \mathfrak{A}_0 \to \mathfrak{Nr}_{\alpha}\mathfrak{B}_0$ and $j_1 : \mathfrak{A}_1 \to \mathfrak{Nr}_{\alpha}\mathfrak{B}_1$ such that for every monomorphism $f : \mathfrak{A}_0 \to \mathfrak{A}_1$ there exists a monomorphism $g : \mathfrak{B}_0 \to \mathfrak{B}_1$ such that $g \circ j_0 = j_1 \circ f \cdot$ We construct our algebras using ultraproducts. Let R be the set of all ordered quadruples $\langle \rho, n, k, l \rangle$ such that: $\rho \in {}^{m}\alpha$ is one to one for some $m \in w$, $n \in w$, k, l are one to one (finite) sequences with

$$k, l \in {}^{n}(\alpha \setminus Rg\rho) \text{ and } Rgk \cap Rgl = \emptyset.$$

For $\rho \in {}^{i}\alpha (i \in w)$ one to one put

$$X_{\rho, n} = \{\langle \sigma, m, k, l \rangle \in R : \rho \subseteq \sigma \text{ and } n \leq m\}.$$

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[1] This implies that the lattice of ideals of $\mathfrak{B}$ is isomorphic to that of $\mathfrak{A}$, via $I \mapsto (I \cap A)$.

It is straightforward to check that the set consisting of all the $X_{\rho, n}$'s is closed under finite intersections. Accordingly, we let M be the proper filter of $\wp(R)$ generated by the $X_{\rho, n}$'s so that

$$M = \{Y \subseteq R : X_{\rho, n} \subseteq Y \text{ for some } \rho \text{ and } n \in w\}.$$

For each $\langle \rho, n, k, l \rangle \in R$, choose a bijection $t(\langle \rho, n, k, l \rangle)$ from $\alpha + w$ onto α such that

$$t(\langle \rho, n, k, l \rangle) \upharpoonright Rg\rho \subseteq Id,$$

and

$$t(\langle \rho, n, k, l \rangle)(a + j) = k_j, \text{ for each } j < n.$$

Now fix $i \in \{0, 1\}$. Let

$$\mathbf{F}(\mathfrak{A}_i) = \prod_{\phi \in R} \mathfrak{Rd}^{t(\phi)}\mathfrak{A}_i \,/\, M.$$

Here $\mathfrak{Rd}^{t(\phi)}\mathfrak{A}_i$ - the $t(\phi)$ reduct of $\mathfrak{A}_i$ - is a $\mathbf{K}_{\alpha+w}$, and so $\mathbf{F}(\mathfrak{A}_i)$- an ultraproduct of these - is also a $\mathbf{K}_{\alpha+w}$. Note too, that for each $\phi \in R$, the algebra $\mathfrak{Rd}^{t\phi}\mathfrak{A}_i$ has universe A_i. Consider a non-zero $x \in A_i$ and $\langle \rho, n, k, l \rangle \in R$. For each $p < n$ for each l_p and each k_p, choose σ such that σ is one to one $\rho \subseteq \sigma$, and $h_{k_p}^{l_p}$ to be in $\mathfrak{Rd}_k^{\rho}\mathfrak{A}$, such that $c_{\sigma_u} \text{o}\, h_{k_p}^{l_p}$ $= h_{k_p}^{l_p}$, whenever, $k_p \leq \mu < k_p + l_p$ and $h_{k_p}^{l_p}(x) \neq 0$. Let j_i be the function from $\mathfrak{A}_i$ into $\mathbf{F}(\mathfrak{A}_i)$ defined to be 0 at 0 and for $x \neq 0$ by,

$$j_i x = \langle (h_{l_0}^{k_0})^{\mathfrak{A}_i} \text{o} \ldots (h_{l_{n-1}}^{k_{n-1}})^{\mathfrak{A}_i} x : \langle \rho, n, k, l \rangle \in R \rangle / M.$$

Since, $h_{l_0}^{k_0} \text{o} \ldots \text{o} h_{l_{n-1}}^{k_{n-1}}$ is a boolean homomorphism from $\mathfrak{A}_i$ into $\mathfrak{Rd}^{t(\langle \rho, n, k, l \rangle)}\mathfrak{A}$, whenever, $\langle \rho, n, k, l \rangle \in R$, then j_i is a Boolean homomorphim from $\mathfrak{A}$ into $\mathbf{F}(\mathfrak{A}_i)$. Consider and $\eta < \alpha$. Then for each $\langle \rho, n, k, l \rangle \in R$ such that $\eta \in Rg\rho$, we have

$$h_{l_0}^{k_0} \circ \ldots h_{l_{n-1}}^{k_{n-1}} \circ c_\eta x = c_\eta \circ h_{l_0}^{k_0} \circ \ldots h_{l_{n-1}}^{k_{n-1}} x,$$

since $\eta \notin Rgk \cup Rgl.$ From the fact that

$$\{\langle \rho, n, k, l\rangle \in R : \eta \in Rg\rho\} \in M,$$

we obtain that f preserves cylindrifications. Assume $i, j \in \alpha.$ Then

$$f(d_{ij}) = \langle h_{l_0}^{k_0} \circ \ldots h_{l_{n-1}}^{k_{n-1}} d_{ij} : \langle \rho, n, k, l\rangle \in R\},$$

but

$$\{\langle \rho, n, k, l\rangle \in R : h_{l_0}^{k_0} \circ \ldots h_{l_{n-1}}^{k_{n-1}} d_{ij} = d_{ij}\} \in M,$$

we obtain that f preserves diagonal elements. Preservation of substitutions is similar. Now consider any $\mu \in (\alpha + w) \sim \alpha.$ Then for any $\langle \rho, n, k, l\rangle \in R$ such that $\mu < \alpha + n,$ we have

$$c_{t\langle \rho, n, k, l\rangle} \circ h_{l_0}^{k_0} \circ \ldots h_{l_{n-1}}^{k_{n-1}} = h_{l_0}^{k_0} \circ \ldots h_{l_{n-1}}^{k_{n-1}}.$$

This equality follows since $(t\langle \rho, n, k, l\rangle)\mu \in Rgk \sim Rgl.$ But

$$\{\langle \rho, n, k, l\rangle \in R, \mu < \alpha + n\} \in M,$$

we see that $f(A_i) \subseteq \mathfrak{Nr}_\alpha \mathbf{F}(\mathfrak{A}_i).$ Consider any $x \in A$ such that $x \neq 0.$ Then for any $\rho \in {}^i\alpha$ and any $n < w,$ there exist sequences $k, l \in {}^n(\alpha \sim Rg\rho)$ one to one such that $Rgk \cap Rgl = \emptyset$ and

$$h_{l_0}^{k_0} \circ \ldots h_{l_{n-1}}^{k_{n-1}} x \neq 0.$$

This follows by a simple inductive argument. Then $j_i \in Ism(\mathfrak{A}_i, \mathfrak{Nr}_\alpha \mathbf{F}(\mathfrak{A}_i)).$ Let g be the function from $\mathbf{F}(\mathfrak{A}_0)$ into $\mathbf{F}(\mathfrak{A}_1)$ defined by:

$$g(\langle x_\phi : \phi \in R\rangle / M) = \langle fx_\phi : \phi \in R\rangle / M.$$

Then it is not hard to check that g is well defined and it is the desired "lifting" function. Now, we show that L has AP. Let $\mathfrak{C}, \mathfrak{A}, \mathfrak{B} \in L.$ Let

$f : \mathfrak{C} \to \mathfrak{A}$ and $g : \mathfrak{C} \to \mathfrak{B}$ be monomorphisms. Then there exist $\mathfrak{A}^+$, $\mathfrak{B}^+$, $\mathfrak{C}^+ \in \mathbf{K}_{\alpha+w}$, embeddings $e_A : \mathfrak{A} \to \mathfrak{Nr}_\alpha \mathfrak{A}^+, e_B : \mathfrak{B} \to \mathfrak{Nr}_\alpha \mathfrak{B}^+, e_C : \mathfrak{C} \to \mathfrak{Nr}_\alpha \mathfrak{C}^+, \bar{f} : \mathfrak{C}^+ \to \mathfrak{A}^+$ and $\bar{g} : \mathfrak{C}^+ \to \mathfrak{B}^+$ such that $\bar{f} \circ e_A = e_A \circ f$ and $\bar{g} \circ e_B = e_B \circ g$. We can assume that $\mathfrak{Sg}^{\mathfrak{A}^+} e_A(A) = \mathfrak{A}^+$ and similarly for $\mathfrak{B}^+$ and $\mathfrak{C}^+$. (Here we are assuming that $\bar{f}[\mathfrak{Sg}^{\mathfrak{C}^+} e_C(C)] \subseteq \mathfrak{Sg}^{\mathfrak{A}^+}(e_A(A))$ and that $\bar{g}[\mathfrak{Sg}^{\mathfrak{C}^+} e_C(C)] \subseteq \mathfrak{Sg}^{\mathfrak{B}^+}(e_B(B).)$ Let $K = \{A \in \mathbf{K}_{\alpha+w} : \mathfrak{A} = \mathfrak{Sg}^{\mathfrak{A}} \mathfrak{Nr}_\alpha \mathfrak{A}\}$. Then by [8] K has *SUPAP*, hence there is a $\mathfrak{D}^+$ in K and $k : \mathfrak{A}^+ \to \mathfrak{D}^+$ and $h : \mathfrak{B}^+ \to \mathfrak{D}^+$ such that $k \circ \bar{f} = h \circ \bar{g}$. Then $k \circ e_A : \mathfrak{A} \to \mathfrak{Nr}_\alpha \mathfrak{D}^+$ and $h \circ e_B : \mathfrak{B} \to \mathfrak{Nr}_\alpha \mathfrak{D}^+$ are one to one and $k \circ e_A \circ f = h \circ e_B \circ g$. For (1) h_k^l can be taken to be s_k^l.

Assume (3). Let $\beta = \alpha + w$. We first prove the following condition $(^{**})$: For $\mathfrak{A}, \mathfrak{A}' \in L$, $\mathfrak{B}, \mathfrak{B}' \in \mathbf{K}_\beta$, e_A, $e_{A'}$ embeddings from $\mathfrak{A}, \mathfrak{A}'$ into $\mathfrak{Nr}_\alpha \mathfrak{B}, \mathfrak{Nr}_\alpha \mathfrak{B}'$, respectively, such that $\mathfrak{Sg}^{\mathfrak{B}}(e_A(A)) = \mathfrak{B}$ and $\mathfrak{Sg}^{\mathfrak{B}'} e_{A'}((A')) = \mathfrak{B}'$, and $i : \mathfrak{A} \to \mathfrak{A}'$ an isomorphism, there exists an isomorphism $\bar{i} : \mathfrak{B} \to \mathfrak{B}'$ such that $\bar{i} \circ e_A = e_{A'} \circ i$. Let $\mu = |A|$. Let x be a bijection from μ onto A that satisfies the premise of (3)(a). Let y be a bijection from μ onto A', such that $i(x_j) = y_j$ for all $j < \mu$. Let $\rho = \langle \Delta^{(\mathfrak{A})} x_j : j < \mu \rangle$, $\mathfrak{D} = \mathfrak{Fr}_\mu^{(\rho)} \mathbf{K}_\beta$, $g_\xi = \xi / Cr_\mu^{(\rho)} \mathbf{K}_\beta$ for all $\xi < \mu$ and $\mathfrak{C} = \mathfrak{Sg}^{\mathfrak{Rd}_\alpha \mathfrak{D}}\{g_\xi : \xi < \mu\}$. Then $\mathfrak{C} \subseteq \mathfrak{Nr}_\alpha \mathfrak{D}$, C generates $\mathfrak{D}$ and by hypothesis $\mathfrak{C} \in L$. There exist $f \in Hom(\mathfrak{D}, \mathfrak{B})$ and $f' \in Hom(\mathfrak{D}, \mathfrak{B}')$ such that $f(g_\xi) = e_A(x_\xi)$ and $f'(g_\xi) = e_{A'}(y_\xi)$ for all $\xi < \mu$. Note that f and f' are both onto. We now have, $e_A \circ i^{-1} \circ e_{A'}^{-1} \circ (f' \upharpoonright \mathfrak{C}) = f \upharpoonright \mathfrak{C}$. Therefore, $Ker\, f' \cap \mathfrak{C} = Ker\, f \cap \mathfrak{C}$. Hence, by (3)($b$) $\mathfrak{Ig}(Ker\, f' \cap \mathfrak{C}) = \mathfrak{Ig}(Ker\, f \cap \mathfrak{C})$. So, $Ker\, f' = Ker\, f$. Let $y \in B$, then there exists $x \in D$ such that $y = f(x)$. Define $\hat{i}(y) = f'(x)$. The map is well defined and is as required. Let $\mathfrak{C} \in L$. let $\mathfrak{A}, \mathfrak{B} \in \mathbf{RK}_\alpha$.

Let $f : \mathfrak{C} \to \mathfrak{A}$ and $g : \mathfrak{C} \to \mathfrak{B}$ be monomorphisms. Then by the Neat Embedding Theorem, there exist $\mathfrak{A}^+, \mathfrak{B}^+, \mathfrak{C}^+, \in \mathbf{K}_{\alpha+w}$ and embeddings $e_A : \mathfrak{A} \to \mathfrak{Nr}_\alpha\mathfrak{A}^+, e_B : \mathfrak{B} \to \mathfrak{Nr}_\alpha\mathfrak{B}^+$ and $e_C : \mathfrak{C} \to \mathfrak{Nr}_\alpha\mathfrak{C}^+$. We can assume that $\mathfrak{Sg}^{\mathfrak{A}^+}e_A(A) = \mathfrak{A}^+$ and similarly for $\mathfrak{B}^+$ and $\mathfrak{C}^+$. Let $f(C)^+ = \mathfrak{Sg}^{\mathfrak{A}^+}e_A(f(C))$ and $g(C)^+ = \mathfrak{Sg}^{\mathfrak{B}^+}e_B(g(C))$. Then by the above there exist $f : \mathfrak{C}^+ \to f(C)^+$ and $\bar{g} : \mathfrak{C}^+ \to g(C)^+$ such that $(e_A \restriction f(C)) \circ f = \bar{f} \circ e_C$ and $(e_B \restriction g(C)) \circ g = \bar{g} \circ e_C$. Now K as defined above has *SUPAP*, hence there is a $\mathfrak{D}^+$ in K and $k : \mathfrak{A}^+ \to \mathfrak{D}^+$ and $h : \mathfrak{B}^+ \to \mathfrak{D}^+$ such that $k \circ \bar{f} = h \circ \bar{g}$. Then $k \circ e_A : \mathfrak{A} \to \mathfrak{Nr}_\alpha\mathfrak{D}^+$ and $h \circ e_B : \mathfrak{B} \to \mathfrak{Nr}_\alpha\mathfrak{D}^+$ are one to one and $k \circ e_A \circ f = h \circ e_B \circ g$. ■

Consider the following condition:

(*) For all $\mathfrak{A} \in L$, whenever, $\mathfrak{B} \in \mathbf{K}_{\alpha+w}$ such that $\mathfrak{A} \subseteq \mathfrak{Nr}_\alpha\mathfrak{B}$, then for all $X \subseteq \mathfrak{A}$, $\mathfrak{Sg}^{\mathfrak{A}}X = \mathfrak{Nr}_\alpha\mathfrak{Sg}^{\mathfrak{B}}X$.

Note that $\mathfrak{Sg}^{\mathfrak{A}}X = \mathfrak{Sg}^{\mathfrak{Nr}_\alpha\mathfrak{B}}X$. hence the above condition states that forming subalgebras commutes with taking neat reducts. (More succinctly: The subalgebra of the neat reduct is the same as the neat reduct of the subalgebra).

Theorem 4. (1) *If L satisfies* (*) *and* [(1) *or* (2) *in Theorem* 3], *then L has SUPAP with respect to* $\mathbf{RK}_\alpha$ *over L.*

(2) *If L satisfies* (*) *and* (3) *in Theorem* 3, *then* $\mathbf{RK}_\alpha$ *has SUPAP with respect to* $\mathbf{RK}_\alpha$ *over L.*

Proof. We only prove (2). The proof of (1) is completely analogous. Assume (*) and (3). Repeating the above proof for *AP*, we have for $\mathfrak{C} \in L$, $\mathfrak{A}, \mathfrak{B} \in \mathbf{RK}_\alpha f : \mathfrak{C} \to \mathfrak{A}$ $g : \mathfrak{C} \to \mathfrak{B}$ monomorphisms, there is a $\mathfrak{D} \in \mathfrak{Nr}_\alpha\mathbf{K}_{\alpha+w}$ and $m : \mathfrak{A} \to \mathfrak{D}\, n : \mathfrak{B} \to \mathfrak{D}$ such that $m \circ f = n \circ g$. Here $m = k \circ e_A$ and $n = h \circ e_B$ with k and h are as above. Denote k by m^+ and h

by n^+. Now we further want to show that if $m(a) \le n(b)$, for $a \in A$ and $b \in B$, then there exists $t \in C$ such that $a \le f(t)$ and $g(t) \le b$. So, let a and b be as indicated . We have $m^+ \circ e_A(a) \le n^+ \circ e_B(b)$, so $m^+(e_A(a)) \le n^+(e_B(b))$. Since L has *SUPAP*, there exist $z \in C^+$ such that $e_A(a) \le \bar{f}(z)$ and $\bar{g}(z) \le e_B(b)$. Let $\Gamma = \Delta z \setminus \alpha$ and $z' = c_{(\Gamma)} z$. (Note that Γ is finite.) So, we obtain that $e_A(c_{(\Gamma)} a) \le \bar{f}(c_{(\Gamma)} z)$ and $\bar{g}(c_{(\Gamma)} z) \le e_B(c_{(\Gamma)} b)$. It follows that $e_A(a) \le \bar{f}(z')$ and $\bar{g}(z') \le e_B(b)$. Now by hypothesis

$$z' \in \mathfrak{Nr}_\alpha \mathfrak{C}^+ = \mathfrak{Sg}^{\mathfrak{Nr}_\alpha \mathfrak{C}^+}(e_C(C)) = e_C(C).$$

So, there exists $t \in C$ with $z' = e_C(t)$. Then we get $e_A(a) \le f'(e_C(t))$ and $\bar{g}(e_C(t)) \le e_B(b)$. It follows that $e_A(a) \le e_A \circ f(t)$ and $e_B \circ g(t) \le e_B(b)$. Hence, $a \le f(t)$ and $g(t) \le b$. We are done. ■

New Consequences of the Above Theorems

• The classes $\mathbf{RK}_\alpha$ does not satisfy (*), since these classes satisfy 3(a) in Theorem 3 but fails to have *SAP* [8]. The **CA** part answers a question of Henkin and Monk posed in the introduction of [7] (p. iv item (5)), since failure of (*) can be paraphrased as: There are generating subreducts that are not neat reducts. Compare with Theorem 2.6.67 in [6].

• The classes $\mathbf{RK}_\alpha$ for infinite α, do not satisfy (3)(b) in Theorem 3, since these classes satisfy (3)(a) but fail to have *AP*. The **CA** part confirms an unsettled conjecture of Tarski in [6] cf. *op cit* top of page 426. That is $\mathbf{Dc}_\alpha$ cannot be replaced in Theorem 2.6.71 of [6] by $\mathbf{RCA}_\alpha$ for infinite α.

• Let $\mathbf{DKc}_\alpha = \{\mathfrak{A} \in \mathbf{K}_\alpha : \alpha \sim \Delta x \text{ is infinite for all } x \in A\}$. In [2], it is shown that $\mathbf{DKc}_\alpha$'s and monadic generated $\mathbf{K}$'s satisfy (3) in Theorem 3 and (*). In particular, minimal algebras satisifies (3) and (*). It is not trivial to show that $\mathbf{CA}$'s and $\mathbf{QEA}$'s of positive characteristic $k > 0$ satisfy (1) in Theorem 3 and (*), cf. [6] Theorem 2.6.54. It thus follows from Theorem 3 that $\mathbf{CA}$'s and $\mathbf{QEA}$'s of positive characteristic $k > 0$, and monadic generated $\mathbf{K}_\alpha$'s have *SUPAP*. (This answers questions of Pigozzi for the $\mathbf{CA}$ case, cf. [12] p. 336, since *SUPAP* implies *SAP*.) It is true that by the above the super amalgam is only representable, but if $\mathfrak{A} \subseteq \mathfrak{B}$ and $\mathfrak{A}$ is of positive characteristic, then so is $\mathfrak{B}$. For monadic generated algebras one takes the subalgebra of the super amalgam generated by the images of the two algebras that lie over the base algebra. This gives a super amalgam that is monadic generated.

• Let $\mathbf{SsK}_\alpha$ denote the class of semisimple algebras in $\mathbf{K}_\alpha$, $\mathbf{ReK}_\alpha$ be the class of algebras satisfying (1) in Theorem 3 and $\mathbf{L}$ be the class of algebras satisfying (2) in Theorem 3. Then $\mathbf{DKc}_\alpha \subseteq \mathbf{SsK}_\alpha \subseteq \mathbf{ReK}_\alpha \subseteq \mathbf{L} \subseteq \mathbf{RK}_\alpha$. The first inclusion is easy. For the second inclusion for $\mathbf{CA}$'s cf. [6] Theorem 2.6.50. Generally, let $\mathfrak{A}$ be semisimple. Let x be non zero and $\Gamma \subseteq \alpha$ be finite. Then there is a maximal ideal I of $\mathfrak{A}$ such that $x \notin I$, and so there is a finite subset Δ of α for which $-c_{(\Delta)}x \in I$. Choose $k, l \in \alpha \sim (\Gamma \cup \Delta)$. Assume that $s_k^l x = 0$. Then $c_{(\Delta)} s_k^l x = s_k^l c_{(\Delta)} x = 0$. Hence $-s_k^l c_{(\Delta)} x = 1$. But then $s_k^l - c_{(\Delta)} x = 1$ so $1 \in I$ which is impossible since I is a proper ideal. $\mathbf{ReK}_\alpha \subseteq \mathbf{L}$ is proved in [6], and so is the last inclusion. The latter follows from the Neat Embedding Theorem, namely $S\mathfrak{Nr}_\alpha \mathbf{K}_{\alpha+w} = \mathbf{RK}_\alpha$. Now the inclusions

$$\mathbf{SsK}_\alpha \subseteq \mathbf{ReK}_\alpha \subseteq \mathbf{L}$$

are proper. To see this let $\mathfrak{A}$ be the full set algebra in the space ${}^\alpha 2$. Then clearly $\mathfrak{A} \in \mathbf{ReK}_\alpha$. Also $\mathfrak{A}$ is not semisimple. Indeed, let $X = \{\langle 0 : k < \alpha \rangle\}$. Then X belongs to every maximal ideal of $\mathfrak{A}$. For if $X \notin I$, then there is a finite subset Γ of α such that $\sim c_{(\Gamma)} X \in I$. Choose $k \in \alpha \sim \Gamma$ and let $\phi = \langle 0 : \mu \in \alpha \sim \{k\} \rangle \cup \langle k, 1 \rangle$. Then $\phi \in \sim c_{(\Gamma)} X$, so $\{\phi\} \in I$. But $X \subseteq c_k \{\phi\}$, so $X \in I$ which is impossible. Now $\mathbf{ReK}_\alpha \subset \mathbf{L}$. The following example is taken from [6] and adapted to the cases considered herein. If we take $\mathfrak{A}$ to be the full set algebra in the space ${}^{\alpha}\alpha$, then $s_k^l(Id \mid \alpha) = 0$ for every $k, l < \alpha$. Suppose that ρ is a finite one to one sequence with $Rg\rho \subseteq \alpha$ and $X \subseteq {}^{\alpha}\alpha$, $X \neq 0$. Let $k \in \alpha \setminus Rg\rho$ and choose $\tau \in {}^{\alpha}\alpha$ such that $k \notin Rg\rho$, $\tau \upharpoonright Rg\rho \subseteq Id$ and τ is one to one. Let

$$h(Y) = \{\phi \in {}^{\alpha}\alpha : \phi \circ \tau \in Y\}.$$

Then it is not hard to show that h satisfies the conclusion of (2) in Theorem 3.

• Now for the main result of this paper. Let α be an infinite ordinal. The class $\mathbf{L}$ as defined in the second item of Theorem 3 does not coincide with the class of representable algebras since it has *AP* with respect to $\mathbf{RK}_\alpha$, while $\mathbf{RK}_\alpha$ fails to have *AP*. The fact that $\mathbf{RCA}_\alpha$ does not have *AP* is proved by Pigozzi [12] and the same result holds for the other algebras considered herein [3]. This answers a question of Henkin Monk and Tarski [6] p.417, formulated as Problem 2.13 in [6]. The latter is one of the very few remaining open questions in the monograph [6].

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